

# A Concept of Explaining the Properties of Elementary Particles in Terms of Manifolds

Julian Ławrynowicz

Institute of Mathematics, Polish Academy of Sciences, The Łódź Branch  
and

Leszek Wojtczak

Institute of Physics, University of Łódź

(Z. Naturforsch. **29 a**, 1407—1417 [1974] ; received May 10, 1974)

The authors propose to explain the magnetic moment of elementary particles by a suitable choice of one pseudo-riemannian manifold — the space of observations — and two general Riemannian manifolds — the spaces of the particle connected with the external electromagnetic and nuclear fields, respectively. By a general Riemannian manifold the authors understand a Riemannian manifold whose associated tensor field is allowed to be degenerate. In this way the mass of a particle as well as its electromagnetic and nuclear properties are determined by means of manifolds and mappings between the corresponding Hilbert spaces. A nuclear reaction is then to be interpreted as a mapping between the corresponding pseudo-riemannian manifolds and the associated general Riemannian manifolds.

The proposal, competitive to the quantum field theory, presents a different way of describing the properties of physical objects. At the moment it is difficult to decide whether this proposal will lead to a satisfactory explanation of more physical phenomena than those explained by means of the quantum field theory, since it needs further research.

## 1. Introduction and Outline of Results

This paper aims at proposing a concept for explaining the properties of elementary particles in terms of topological and metrical properties of pseudo-riemannian and general Riemannian manifolds.

By a *pseudo-riemannian* manifold we mean a  $C^\infty$ -differentiable paracompact (cf. e.g. Ref. <sup>1</sup>) connected manifold endowed with a pseudo-riemannian metric, i.e. a symmetric  $C^\infty$  tensor field of type (0,2) which is nondegenerate and has at each point the same index, different from zero and from the dimension of the manifold. If not otherwise stated, this dimension is supposed to be 4 and the index 1.

If in the above definition we let the index to be 0, i.e. the metric to be positive definite, we call this metric *Riemannian* and the corresponding manifold — a Riemannian manifold. If in the definition of a Riemannian manifold we reject the assumption of connectedness and allow the metric to be degenerate, we call this metric *general Riemannian* and the corresponding manifold — a general Riemannian manifold.

When choosing the manifolds in question the authors follow the lines of von Westenholz <sup>2</sup> and generalize his ideas which have a background in Misner and Wheeler's ideas <sup>3</sup> of explaining physical quantities in terms of pure geometry but without adding them to it. This leads of course to an important consequence, namely, that we have to replace the Dirac equation by another (Dirac-like) equation and this concept has its background in a recent paper of Dirac <sup>4</sup>.

We suppose that the elementary particle in question is connected with two four-dimensional general Riemannian manifolds — the spaces of the particle, assigned to the external electromagnetic and nuclear fields, respectively, by means of the electromagnetic field four-tensor and the nuclear field four-pseudotensor (for the definition and properties of pseudotensors cf. e.g. Ref. <sup>5</sup>). More exactly, both manifolds are supposed to be manifolds with boundary (cf. e.g. Ref. <sup>6</sup>). These boundaries play here the part of equipotential hypersurfaces corresponding to the electromagnetic and nuclear fields, respectively.

The general Riemannian manifolds in question are shown to have some duality properties that will perhaps lead in future to the construction of an *almost complex manifold* (cf. e.g. Ref. <sup>7</sup>) endowed with an *hermitian structure*, generated by the manifolds in question:  $N_e$  and  $N_n$ , say.

Reprint requests to Prof. Julian Ławrynowicz, ul. Piotrkowska 82 m. 60, PL-90-102 Łódź, Poland.



Dieses Werk wurde im Jahr 2013 vom Verlag Zeitschrift für Naturforschung in Zusammenarbeit mit der Max-Planck-Gesellschaft zur Förderung der Wissenschaften e.V. digitalisiert und unter folgender Lizenz veröffentlicht: Creative Commons Namensnennung-Keine Bearbeitung 3.0 Deutschland Lizenz.

Zum 01.01.2015 ist eine Anpassung der Lizenzbedingungen (Entfall der Creative Commons Lizenzbedingung „Keine Bearbeitung“) beabsichtigt, um eine Nachnutzung auch im Rahmen zukünftiger wissenschaftlicher Nutzungsformen zu ermöglichen.

This work has been digitalized and published in 2013 by Verlag Zeitschrift für Naturforschung in cooperation with the Max Planck Society for the Advancement of Science under a Creative Commons Attribution-NoDerivs 3.0 Germany License.

On 01.01.2015 it is planned to change the License Conditions (the removal of the Creative Commons License condition “no derivative works”). This is to allow reuse in the area of future scientific usage.

In order to give an example of such a construction let us denote by  $\text{top } \mathbf{N}_e$  and  $\text{top } \mathbf{N}_n$  the corresponding paracompact topological spaces, and by  $g_e'$  and  $g_n'$ —their general Riemannian metrics. Let  $\mathbf{L}$  be the Cartesian product of  $\text{top } \mathbf{N}_e$  and  $\text{top } \mathbf{N}_n$ , endowed with the natural maximal atlas  $\text{atl } \mathbf{L}$ :

$$\mathbf{L} = (\text{top } \mathbf{L}, \text{atl } \mathbf{L}), \text{top } \mathbf{L} = \text{top } \mathbf{N}_e \times \text{top } \mathbf{N}_n. \quad (1.1)$$

When introducing an hermitian structure on  $\mathbf{L}$  we have to suppose that  $\mathbf{L}$  admits an *almost complex structure*, i. e. that there exists a  $C^\infty$  field  $J$  of endomorphisms  $J_z$  of the tangent spaces  $T_z \mathbf{L}$  at points  $z$  of  $\mathbf{L}$  such that the composed mapping  $-J_z \circ J_z$  is the identity mapping. For any  $z = (x, y)$ , where  $x$  and  $y$  are points of  $\mathbf{N}_e$  and  $\mathbf{N}_n$ , respectively, we denote by  $\text{pr}_z^e$  the mapping induced by the projection  $z \mapsto x$  between the corresponding tangent spaces:  $\text{pr}_z^e: T_z \mathbf{L} \rightarrow T_x \mathbf{N}_e$  and, analogously, we introduce  $\text{pr}_z^n: T_z \mathbf{L} \rightarrow T_y \mathbf{N}_n$ . Next, for any two vectors  $\mathbf{v}$  and  $\mathbf{w}$  of  $T_z \mathbf{L}$ , let

$$\hat{g}(\mathbf{v}, \mathbf{w}) = g_e'(\text{pr}_z^e \mathbf{v}, \text{pr}_z^e \mathbf{w}) + g_n'(\text{pr}_z^n \mathbf{v}, \text{pr}_z^n \mathbf{w}) \quad (1.2)$$

and, finally,

$$h(\mathbf{v}, \mathbf{w}) = \hat{g}(\mathbf{v}, \mathbf{w}) - i \hat{g}(J \mathbf{v}, \mathbf{w}) + i \hat{g}(\mathbf{v}, J \mathbf{w}) + \hat{g}(J \mathbf{v}, J \mathbf{w}), \quad (1.3)$$

where  $i$  denotes the imaginary unit:  $i^2 = -1$ . It is easy to show that  $(\mathbf{L}, J, h)$ , defined by formulae (1.1) and (1.3), where  $J$  is an almost complex structure of the tangent bundle  $T \mathbf{L} = \bigcup T_z \mathbf{L}$ , is an example of the required manifold.

It is a separate problem whether this manifold, or another almost complex manifold generated by  $\mathbf{N}_e$  and  $\mathbf{N}_n$ , admits a *complex (analytic) structure*. Besides it would be natural to introduce some related complex field tensor and other related quantities. At this stage, however, we prefer a real space description.

Now, by means of some mappings from the Hilbert spaces assigned in a natural way to the manifolds  $\mathbf{N}_e$  and  $\mathbf{N}_n$  we generate a third manifold  $\mathbf{M}$  — pseudo-riemannian, which corresponds to the space of observations, and determine the electromagnetic and nuclear properties of the particle or, more generally, of a system of particles, by means of the topology of  $\mathbf{M}$ , while the mass, magnetic moment, and other properties — by means of the pseudo-riemannian metric  $g$  of  $\mathbf{M}$ . The elementarity of the particle means that it represents a “hole” in  $\mathbf{M}$ , i. e. remains beyond admissible places of observation. It is respected that the manifolds  $\mathbf{M}$ ,

$\mathbf{N}_e$ , and  $\mathbf{N}_n$  can be embedded in the five-dimensional Euclidean space (cf. Ref. <sup>8</sup>).

The paper is concluded by the statement of a possibility to explain nuclear reactions by seeking selection rules for composition of the obtained manifolds as well as by choosing mappings which transform the system in question before a reaction onto the system after this reaction. Since the manifold  $\mathbf{M}$  corresponds to the space of observation, it should stay unchanged after a nuclear reaction. This concerns in particular the light cones and suggests the use of conformal mappings. In the case where an external particle is introduced, the light cones change in a bounded way according to a finite velocity of this particle and this suggests the use of quasiconformal mappings <sup>9</sup>.

The proposal presented in this paper is not the first trial of explaining the properties of elementary particles in terms of suitable manifolds corresponding to those particles. In order to be concise, we confine ourselves to quote a fundamental paper of Dirac as well as two recent papers <sup>10</sup> in this direction, where further references are to be found. The distinction between  $\mathbf{M}$  and  $\mathbf{N}$  is in fact motivated by Dirac's considerations, but it seems that our construction permits to include effectively electromagnetic and nuclear interactions.

## 2. Hamiltonian of a Free Particle in the System of Observations

Let us assign to each elementary particle  $\chi$  some four-dimensional general Riemannian manifold  $\mathbf{N}$  with boundary (cf. Section 1). More exactly, we assign to  $\chi$  a *fibre bundle* <sup>11</sup>  $B_{\mathbf{N}}$ , where  $\mathbf{N}$  is the fibre space. We let the real line  $\mathbf{R}$  to be the base space of  $B_{\mathbf{N}}$  and denote by  $\mathbf{N}_{\#}$  the corresponding typical fibre which is not necessarily connected. We aim to determine the topological structure and the general Riemannian tensor field  $g'$  of  $\mathbf{N}$  so that the operator equations for state vectors, which we take into consideration, would imply the properties of  $\chi$ , which are possible to be noticed in the system of observations.

According to quantum mechanics (cf. e.g. Ref. <sup>5</sup>) the state equation for a free particle has the form

$$H|t\rangle = i\hbar(\partial/\partial t)|t\rangle, \quad (2.1)$$

where  $t$  is in  $\mathbf{R}$  and  $|t\rangle$  is an element of the Hilbert space  $\mathbf{H}(B_{\mathbf{N}})$  assigned to  $\chi$  in the usual way <sup>5</sup>. For

a free particle the energy operator  $H$  has the form  $H = \mathbf{p}^2/2m$  with the standard meaning of  $\mathbf{p}$  and  $m$ , since we assume that in the own system, i.e. the system connected with this particle, the velocity of propagation of interactions is infinite.

We further assume that the system of observations is situated in some four-dimensional pseudo-riemannian manifold  $\mathbf{M}$  (cf. Sect. 1) depending on  $\chi$  and that  $\chi$  represents a "hole" of it, i.e. it remains beyond admissible places of observations. More exactly, we assign to  $\chi$  a fibre bundle  $B_{\mathbf{M}}$ , where  $\mathbf{M}$  is the fibre space. We let  $\mathbf{R}$  to be the base space of  $B_{\mathbf{M}}$  and denote by  $M_{\#}$  the corresponding typical fibre. In the case of absence of particles we take as  $\mathbf{M}$  the Minkowski space-time. The pseudo-riemannian tensor field of  $\mathbf{M}$  will be denoted by  $g$ .

Since we consider as the space of observations the manifold  $\mathbf{M}$ , in order to decide upon the properties of the particle observed in  $\mathbf{M}$ , we have to transform Eq. (2.1) to the system corresponding to  $\mathbf{M}$ . As it should be natural to obtain the results of the standard theory in the limit case when  $\chi$  reduces to a point, it seems proper to assume that each vector of  $\mathbf{H}(B_{\mathbf{N}})$  belongs to  $\mathbf{H}(B_{\mathbf{M}})$  and postulate the mapping  $V$  of  $\mathbf{H}(B_{\mathbf{N}})$  into  $\mathbf{H}(B_{\mathbf{M}})$  of the form

$$V(x, t) = \exp\{(i/\hbar)\mathcal{H}(x, t)\}, \quad (2.2)$$

where  $t$  and  $x$  are points of  $\mathbf{R}$  and  $\mathbf{N}_{\#}$ , respectively, and  $\mathcal{H}: \mathbf{H}(B_{\mathbf{N}}) \rightarrow \mathbf{H}(B_{\mathbf{M}})$ . After the application of the mapping (2.2) to elements  $(t, x)$  in Eq. (2.1), this equation becomes

$$V[H - (\partial/\partial t)\mathcal{H}]V^{-1}(V|t) = i\hbar(\partial/\partial t)(V|t), \quad (2.3)$$

where obviously  $V|t$  are elements of  $\mathbf{H}(B_{\mathbf{M}})$ .

From the physical point of view the mapping  $V$  expresses the action of the particle on its neighbourhood and of the neighbourhood on the particle, realized in standard theories through virtual particles of the field. The expression

$$H = V[H - (\partial/\partial t)\mathcal{H}]V^{-1} \quad (2.4)$$

is to be interpreted as the Hamiltonian describing the action of the particle  $\chi$  in the space of observations  $\mathbf{M}$ . The lowest possible eigenvalue of  $H$  should be equal to the rest energy of  $\chi$ .

A trivial example of a mapping  $V$  is connected with passing from the Riemannian manifold  $\mathbf{N}$  assigned to  $\chi$  with the metric of the four-dimensional

Euclidean space to the pseudo-riemannian manifold  $\mathbf{M}$  assigned to the exterior of  $\chi$  with the metric of the Minkowski space-time. In this case, if  $\text{supp } \mathbf{M}$  denotes the set of points of  $\mathbf{M}$  etc., we have  $\text{supp } \mathbf{M} \cup \text{supp } \mathbf{N} \subset \text{supp } \mathbf{R}^4$ ,  $\text{supp } \mathbf{M} \cap \text{supp } \mathbf{N} = \emptyset$ . Let denote the mapping  $\mathcal{H}$  corresponding to this case by  $\mathcal{H}_0$ . It is worth-while to note that the Hamiltonian  $\mathcal{H}_0(x, t) = \mathcal{H}_0(x)t$  can be chosen in the form which leads to a differential equation of the second degree and a state vector  $V|t\rangle$  in the scalar form (cf. Sect. 8). In the case where  $\mathbf{N}$  reduces to a point, we arrive at the Dirac equation for a free particle in the Minkowski space-time. In case of a particle with nonzero dimensions, in the system of observations the particle is described by the Dirac equation determined for the state vectors in the Hilbert space  $\mathbf{H}(B_{\mathbf{M}})$ .

In the physical interpretation the above case corresponds to a particle without interaction with virtual fields. In case of interactions with such particles  $\mathcal{H}$  is a mapping which induces a transformation of the manifold  $\mathbf{N}$  modified by these virtual particles into the manifold  $\mathbf{M}$  which remains unchanged. Thus, if we set

$$\mathcal{H}(x) = \mathcal{H}_0(x) + \Delta\mathcal{H}(x), \quad x \in \text{supp } \mathbf{N}_{\#},$$

where  $\Delta\mathcal{H}(x)$  denotes the Hamiltonian of interactions with virtual particles, then the Hamiltonian in the space of observations  $\mathbf{M}$  admits the form

$$\mathcal{H}(x') = \exp\{(i/\hbar)\mathcal{H}_0[x(x')]\} \mathcal{H}[x(x')] \cdot \exp\{-(i/\hbar)\mathcal{H}_0[x(x')]\} + U(x'), \quad x' \in \text{supp } \mathbf{M}_{\#},$$

where

$$\begin{aligned} U(x') = & \exp\{(i/\hbar)\mathcal{H}[x(x')]\} \mathcal{H}[x(x')] \\ & \cdot \exp\{-(i/\hbar)\mathcal{H}[x(x')]\} \\ & - \exp\{(i/\hbar)\mathcal{H}_0[x(x')]\} \mathcal{H}[x(x')] \\ & \cdot \exp\{-(i/\hbar)\mathcal{H}_0[x(x')]\}. \end{aligned}$$

Here  $U$  is to be interpreted as the potential of external forces. Therefore the image of a particle in the system of observations is the image of the particle remaining within the field of external forces and the nature of these forces seems to be analogous to the classical case of the *Coriolis forces*.

The process of passing from the own system to the system of observations is connected with the visualisation of the spin which we define — as usually — as the own angular momentum of the particle. In the description proposed here *the introduction of the notion of spin gains a new meaning:*



in the own system the spin does not appear since in this case the particle rests, whereas in the system of observations we may assign to the particle the spin  $s$  as the angular momentum of the rotating particle observed in this system.

Thus, for a given manifold  $\mathbf{N}$  with the properties described above, the mapping  $V$  determines the corresponding manifold  $\mathbf{M}$  for preassigned boundary conditions. If we push aside the problem of determining  $\mathbf{M}$ , then to different particles correspond different mappings  $V$ . It is important to notice that the elementarity of a particle is here understood in the sense that we are not interested in the manifold  $\mathbf{N}$  and we only agree that Eq. (2.1) which describes properties of the particle has the same form for all possible particles. With respect to different transformations the manifold  $\mathbf{M}$  changes according to observation of various particles and thus depends on the particle observed.

If we observe two or more particles within  $\mathbf{M}$ , then this manifold is a consequence of a suitable composition of the corresponding mappings  $V$ . In the physical interpretation to different interactions there correspond different mappings of the topological and metrical structure of  $\mathbf{M}$ .

### 3. A Particle in External Fields: Electromagnetic and Nuclear

Interaction of particles may be expressed by means of external fields where the particle in question is situated. We determine interaction of fields with the help of a function yielded by the metrical properties of the manifold  $\mathbf{N}$ .

Namely, suppose that  $\mathbf{N}$  is in an arbitrary external field applied. In view of the finiteness of the particle and hence in view of the *infinite* velocity of propagation of interactions within  $\mathbf{N}$ , interactions that enter  $\mathbf{N}$  by different points of its boundary reach all points of  $\mathbf{N}$  in the infinitely short time. Hence, in spite of inhomogeneity of this interaction at the boundary, inside  $\mathbf{N}$  the value of this interaction is invariant under changes of position within  $\mathbf{N}$  and equal to the sum of all reaching signals.

Consequently, in the case of electromagnetic as well as nuclear fields, when considering the corresponding manifolds  $\mathbf{N}$ , which to be more precise we will denote by  $\mathbf{N}_e$  and  $\mathbf{N}_n$ , respectively, we may assume that the corresponding external fields can

be expressed as some four-tensors with constant components. Keeping at present the values of these components as parameters, let us notice here that the basic physical fields can be described by means of two four-vectors expressing standard electromagnetic interaction and nuclear interaction determined in an analogous way, respectively.

Owing to the fact that the Hamiltonian, as a physical quantity, has to be invariant under mappings from  $\mathbf{N}_e$  resp.  $\mathbf{N}_n$  into itself, the addend of the Hamiltonian that corresponds to the interaction with the external fields and expresses the potential energy can be represented by means of the scalar products  $g_e'(\mathfrak{A}^e, \mathfrak{v}^e)$  and  $g_n'(\mathfrak{A}^n, \mathfrak{v}^n)$ , where  $\mathfrak{A}^e$  and  $\mathfrak{A}^n$  are constant four-vector potentials corresponding to the electromagnetic and nuclear case, respectively, while  $\mathfrak{v}^e$  and  $\mathfrak{v}^n$  are the four-currents assigned with the finding of the particle only, independently of its properties. Consequently, in accordance with formula (1.2), we obtain

$$H = (1/2 m) \mathbf{p}^2 + g_e'(\mathfrak{A}^e, \mathfrak{v}^e) + g_n'(\mathfrak{A}^n, \mathfrak{v}^n). \quad (3.1)$$

In the space  $\mathbf{M}$  of observations the motion of the particle is described by Eq. (2.3) for the Hamiltonian (3.1). The mapping  $V$  stays here invariant since it describes transition from  $\mathbf{N}_e$  resp.  $\mathbf{N}_n$  to  $\mathbf{M}$  independently of external fields acting within  $\mathbf{N}_e$  resp.  $\mathbf{N}_n$ .

Summing up, in spite of the fact that we consider now a particle in an external field applied, the mapping  $V$  corresponding to that particle stays fixed, while the description of the motion of the particle varies in the form of appearance of the addend  $g_e'(\mathfrak{A}^e, \mathfrak{v}^e) + g_n'(\mathfrak{A}^n, \mathfrak{v}^n)$ . The transformation of the Hamiltonian (3.1) under the mapping  $V$  leads to the Hamiltonian (2.4), where the potential energy is composed of the potential energy of really existing acting fields  $\mathfrak{A}^e$  and  $\mathfrak{A}^n$  and of fictitious fields yielded by the mapping  $V$ .

### 4. Electric Charge and Magnetic Dipole as a Manifestation of Topological Properties of the Manifold Associated with the Space of Observations

We proceed now to the question of determining the manifold  $\mathbf{M}$  so that the electric charges and magnetic dipoles be, under a proper formulation of the problem, uniquely determined by  $\mathbf{M}$ . This question is not new and we may confine ourselves

to give here a suitable interpretation of some results due to Misner and Wheeler<sup>3</sup> and von Westenholz<sup>2</sup>.

Suppose, more generally, that we wish to assign  $\mathbf{M}$  so that it exhibit quantized point electric charges  $e_1, \dots, e_n$  and associated with them magnetic charges  $\mu_1, \dots, \mu_n$  due to monopoles distributed over some points  $\tilde{P}_1, \dots, \tilde{P}_n$  of  $\mathbf{M}$ . Here it is essential to remark that, as pointed out by Misner and Wheeler<sup>3</sup>, magnetic monopoles do not exist in a fully classical geometrical theory, since, if the electromagnetic field is derived from a vector potential, then there is a zero net flux through every closed surface and then, by Stokes' theorem, there is no magnetic charge. If, however, quantized charge is associated with space-time, it should be referred to a semi-classical theory of von Westenholz<sup>2</sup>, where in order to explain that all charges are integral multiples of a unit charge, one has to introduce magnetic monopoles. Thus, according to Dirac<sup>12</sup>, we have to accept the pole strengths of magnetic poles as integral multiples of the Planck constant divided by the unit charge.

Let now  $C_p(\mathbf{M})$  denote the  $p$ -th group of singular chain complexes of  $\mathbf{M}$ , i.e. the collection of all formal sums of real multiples of singular  $p$ -simplexes  $\sigma_j^p$  in  $\mathbf{M}$  endowed with the corresponding group structure. Under a singular  $p$ -simplex in  $\mathbf{M}$  we mean, as usually (cf. e.g. Ref.<sup>13</sup>), a continuous mapping from the standard Euclidean  $p$ -simplex into  $\mathbf{M}$ . Further, let  $\partial$  be the corresponding boundary operator, i.e. the operator assigning to each chain  $\sum \lambda_j \sigma_j^p$  of  $C_p(\mathbf{M})$  the chain  $\sum \lambda_j \partial \sigma_j^p$ , where

$$\partial \sigma_j^p = \sum_{k=0}^p (-1)^k \sigma_j^{p-1} \circ e_p^k,$$

$$e_p^k(e^l) = \begin{cases} e^l & \text{for } l < k \\ e^{l+1} & \text{for } l \geq k \end{cases} \quad (0 \leq k \leq p, p > 0),$$

and  $(e^0, e^1, \dots)$  denotes the Hilbert cube. Next let  $\dot{C}_p(\mathbf{M})$  denote the group of singular  $p$ -cycles in  $\mathbf{M}$ , i.e. singular  $p$ -chain complexes in  $\mathbf{M}$  with the vanishing boundary, and  $\partial C_{p+1}(\mathbf{M})$  the group consisting of all boundaries of some singular  $(p+1)$ -chain complexes in  $\mathbf{M}$ . Finally, owing to an obvious inclusion of  $\partial C_{p+1}(\mathbf{M})$  in  $\dot{C}_p(\mathbf{M})$ , we introduce (cf. e.g. Ref.<sup>13</sup>) the  $p$ -th singular homology group of  $\mathbf{M}$  as the quotient group  $H_p(\mathbf{M}) = \dot{C}_p(\mathbf{M}) / \partial C_{p+1}(\mathbf{M})$ .

In a duality to the above definitions let  $F^p(\mathbf{M})$  denote the real vector space of all differential  $p$ -forms on  $\mathbf{M}$  and  $d$  the exterior derivative operator (cf. e.g. Ref.<sup>14</sup>). Next we let  $\dot{F}^p(\mathbf{M})$  denote the group of all closed forms of  $F^p(\mathbf{M})$ , i.e. forms of  $F^p(\mathbf{M})$  with the vanishing exterior derivative, and  $dF^{p-1}(\mathbf{M})$  the group of all exact forms of  $F^p(\mathbf{M})$ , i.e. the group consisting of all exterior derivatives of some forms of  $F^{p-1}(\mathbf{M})$ . Finally, owing to an obvious inclusion of  $dF^{p-1}(\mathbf{M})$  in  $\dot{F}^p(\mathbf{M})$  we introduce the  $p$ -th de Rham cohomology group of  $\mathbf{M}$  as the quotient group  $H^p(\mathbf{M}) = \dot{F}^p(\mathbf{M}) / dF^{p-1}(\mathbf{M})$ .

Consider next the standard four-dimensional manifold  $\tilde{\mathbf{E}}_e$  with  $\text{supp } \tilde{\mathbf{E}}_e = \text{supp } \mathbf{M} \cup \text{supp } \mathbf{N}_e$ , endowed with the Minkowski metric. According to Misner and Wheeler<sup>3</sup> in order to assure that  $\tilde{\mathbf{E}}_e$  permits unquantized charge it is sufficient to take  $\text{supp } \tilde{\mathbf{E}}_e$  in the form of Cartesian product of the real line and either a  $k$ -pierced three-sphere or a three-torus and — on the other hand — any space  $\tilde{\mathbf{E}}_e$  in question may be supposed to be given in such a form. By a  $k$ -pierced sphere we understand the sphere with  $k$  pairs of nonoverlapping polar caps excluded and the corresponding points of the resulting boundaries of each pair of antipodal caps identified. In order to agree with Sect. 2 we have to drill in the resulting manifold also the additional non-intersecting holes corresponding to the particles in question, so we have to consider the submanifold  $\mathbf{E}$  of  $\tilde{\mathbf{E}}_e$  with  $\text{supp } \mathbf{E} = \text{supp } \mathbf{M}$ .

Suppose now that  $\mathbf{M}$  permits quantized charge, i.e.  $4\pi(e_1 \tilde{P}_1 + \dots + e_n \tilde{P}_n)$  is in  $C_0(\mathbf{M})$  and that  $\mathbf{E}$  permits unquantized charge. Then, according to von Westenholz<sup>2</sup>, there exists a  $C^1$ -diffeomorphism  $\Phi: \mathbf{E} \rightarrow \mathbf{M}$  such that the mapping  $\Phi_*: \dot{C}_2(\mathbf{E}) \rightarrow \dot{C}_2(\mathbf{M})$  induced by  $\Phi$  satisfies the conditions

$$\int_{\Phi_* e_j} \tilde{\omega} = 4\pi e_j, \quad (4.1)$$

where  $\tilde{\omega}$  is in  $F^2(\mathbf{M})$  and corresponds to the electromagnetic field tensor,  $e_j$ ,  $j=1, \dots, \beta_2$ , are the fundamental cycles in  $\dot{C}_2(\mathbf{E})$ , and  $\beta_2$  is the second Betti number of  $\mathbf{E}$ , i.e. the number of such cycles. Here we have to suppose that  $n < \beta_2$ . Conversely, let  $\mathbf{M}$  be associated with an  $\mathbf{E}$  which permits unquantized charges by means of a  $C^1$ -diffeomorphism  $\Phi: \mathbf{E} \rightarrow \mathbf{M}$ . Then Eq. (4.1) is a necessary condition for  $\mathbf{M}$  to exhibit quantized charges as a manifestation of its topology.

It is clear that Eq. (4.1) may be considered as a *geometrical definition of a quantized charge in  $\mathbf{M}$* . In order to do so with magnetic charge we write in local coordinates  $\tilde{\omega} = F_{jk} dx^j dx^k$ , where  $F_{jk}$  represents the electromagnetic field four-tensor, and define  $\tilde{\omega}^*$  as the form which can be written in the same local coordinates as  $\tilde{\omega}^* = *F_{jk} dx^j dx^k$ , where  $*F_{jk}$  represents the dual of  $F_{jk}$ . Consequently

$$\int_{\Phi^* c_j} \tilde{\omega}^* = 4\pi \mu_j \quad (4.2)$$

give the required formulae.

Finally, let  $\gamma$  be the charge density in  $\mathbf{M}$ , i. e.

$$\gamma = (1/c) (i_1 dx^2 \wedge dx^3 + i_2 dx^3 \wedge dx^1 + i_3 dx^1 \wedge dx^2) \wedge dx^0 - \varrho dx^1 \wedge dx^2 \wedge dx^3, \quad (4.3)$$

where the vector  $\mathbf{i} = (i_1, i_2, i_3)$  and the scalar  $\varrho$  satisfy the continuity equation

$$\operatorname{div} \mathbf{i} + (\partial/\partial x^0) \varrho = 0,$$

while  $\wedge$  denotes the exterior product operation (cf. e. g. Ref. <sup>14</sup>). As observed by von Westenholz <sup>2</sup>, *another important necessary condition for  $\mathbf{M}$  to exhibit quantized charges as a manifestation of its topology is that if  $\gamma$  is the charge density (4.3) in  $\mathbf{M}$ , then  $\gamma$  is exact*, i. e. there is a form  $\beta$  of  $F^2(\mathbf{M})$  such that  $\gamma = d\beta$ . In other words the subgroup  $H_\gamma^3(\mathbf{M})$  of the third de Rham cohomology group  $H^3(\mathbf{M})$ , consisting of elements generated by the density (4.3), contains one element only:

$$H_\gamma^3(\mathbf{M}) = 0. \quad (4.4)$$

The quoted paper <sup>2</sup> gives also a sufficient condition in terms of homotopy, but we do not apply it here since it is probably too strong, i. e. not necessary.

## 5. Nuclear Charge and Nuclear Dipole as a Manifestation of Topological Properties of the Manifold Associated with the Space of Observations

We turn now our attention to the problem of determining the manifold  $\mathbf{M}$  in question so that the nuclear charges and nuclear dipoles be, under a proper formulation of the problem, uniquely determined by  $\mathbf{M}$ . Arguing as in the case of electromagnetic four-current we arrive at the results which may be formulated as follows.

Consider the standard four-dimensional manifold  $\tilde{\mathbf{E}}_n$  with  $\operatorname{supp} \tilde{\mathbf{E}}_n = \operatorname{supp} \mathbf{M} \cup \operatorname{supp} \mathbf{N}_n$ , endowed with the Minkowski metric. It is clear that in order to assure that  $\mathbf{E}_n$  permits unquantized nuclear

charge it is sufficient to take, as before,  $\operatorname{supp} \tilde{\mathbf{E}}_n$  in the form of Cartesian product of the real line and either a  $k$ -pierced three-sphere or a three-torus and — on the other hand — any space  $\tilde{\mathbf{E}}_n$  in question may be supposed to be given in such a form. In order to agree with Sect. 2 we have to drill in the resulting manifold also the additional nonintersecting holes corresponding to the particles in question, so we have to consider the submanifold  $\mathbf{E}$  of  $\tilde{\mathbf{E}}_n$  with  $\operatorname{supp} \mathbf{E} = \operatorname{supp} \mathbf{M}$ .

In this way, since — in our interpretation — the manifold  $\mathbf{N}$ , discussed in Sect. 2, is modified by virtual particles (cf. Sect. 2) so that we are led to two manifolds  $\mathbf{N}_e$  and  $\mathbf{N}_n$  (cf. Sect. 3), while the manifold  $\mathbf{M}$  remains unchanged (cf. Sect. 2), we are also led to two corresponding manifolds  $\tilde{\mathbf{E}}_e$  and  $\tilde{\mathbf{E}}_n$ , while the manifold  $\mathbf{E}$  is the same in both cases.

Suppose now that  $\mathbf{M}$  permits quantized nuclear charge, i. e.  $4\pi(g_1 \tilde{Q}_1 + \dots + g_m \tilde{Q}_m)$  is in  $C_0(\mathbf{M})$  and that  $\mathbf{E}$  permits unquantized nuclear charge. Here  $g_1, \dots, g_m$  are quantized point nuclear charges, while  $\tilde{Q}_1, \dots, \tilde{Q}_m$  are the points of  $\mathbf{M}$  over which there are distributed some charges  $r_1, \dots, r_m$  of nuclear monopoles associated with  $g_1, \dots, g_m$ , respectively. Then there exists a  $C^1$ -diffeomorphism  $\Psi: \mathbf{E} \rightarrow \mathbf{M}$  such that the mapping  $\Psi_*: \hat{C}_2(\mathbf{E}) \rightarrow \hat{C}_2(\mathbf{M})$  induced by  $\Psi$  satisfies the conditions

$$\int_{\Psi_* c_j} \tilde{\varphi} = 4\pi g_j, \quad (5.1)$$

where  $\tilde{\varphi}$  is in  $F^2(\mathbf{M})$  and corresponds to the nuclear field four-pseudotensor (for the definition and properties of pseudotensors cf. e. g. Ref. <sup>5</sup>), and  $c_j, j=1, \dots, \beta_2$ , are — as before — the fundamental cycles in  $C_2(\mathbf{E})$ . Here we have to suppose that  $m < \beta_2$ . Conversely, let  $\mathbf{M}$  be associated with an  $\mathbf{E}$  which permits unquantized nuclear charges by means of a  $C^1$ -diffeomorphism  $\Psi: \mathbf{E} \rightarrow \mathbf{M}$ . Then Eq. (5.1) is a necessary condition for  $\mathbf{M}$  to exhibit quantized nuclear charges as a manifestation of its topology.

Analogously, since in local coordinates  $\tilde{\varphi} = G_{jk} dx^j dx^k$ , where  $G_{jk}$  represents the nuclear field four-pseudotensor, we may define  $\tilde{\varphi}^*$  as the form which can be written in the same local coordinates as  $\tilde{\varphi}^* = *G_{jk} dx^j dx^k$ , where  $*G_{jk}$  represents the dual of  $G_{jk}$ . Then

$$\int_{\Psi_* c_j} \tilde{\varphi}^* + 4\pi r_j \quad (5.2)$$

give the required formulae for the charges of nuclear monopoles.

It seems that  $G_{jk}$  cannot be, in general, constant since the corresponding mapping from  $F^2(\mathbf{N}_n)$  into  $F^2(\mathbf{M})$  has to change the vector structure of  $\mathfrak{A}^n$  to to the pseudovector structure of the corresponding potential  $\mathfrak{A}_n$  in  $\mathbf{M}$ .

Finally, let  $\tau$  be the nuclear charge density in  $\mathbf{M}$ , i. e.

$$\tau = (1/c) (h_1 dx^2 \wedge dx^3 + h_2 dx^3 \wedge dx^1 + h_3 dx^1 \wedge dx^2) \wedge dx^0 - \vartheta dx^1 \wedge dx^2 \wedge dx^3, \quad (5.3)$$

where the pseudovector (cf. e. g. Ref. <sup>5</sup>)  $\mathbf{h} = (h_1, h_2, h_3)$  and the pseudoscalar  $\vartheta$  satisfy the continuity equation

$$\operatorname{div} \mathbf{h} + (\partial/\partial x^0) \vartheta = 0.$$

Then the counterpart of the condition (4.4) reads:

$$H_\tau^3(\mathbf{M}) = 0. \quad (5.4)$$

Thus we conclude the present section with the statement that in our interpretation the electromagnetic and nuclear fields are connected with the corresponding magnetic and nuclear monopoles situated, in general, at different points  $\tilde{P}_1, \dots, \tilde{P}_n$  and  $\tilde{Q}_1, \dots, \tilde{Q}_m$ , respectively. The difference between the interactions in question is thus yielded by the topological differences between the corresponding manifolds  $\mathbf{N}_e$  and  $\mathbf{N}_n$ . From the physical point of view this difference plays the part of virtual particles in the sense of conventional theories. Furthermore, we remark that according to our concept the effective interaction within  $\mathbf{M}$  is not directly determined by the nuclear four-current  $(\mathbf{h}, \vartheta)$ , but by a mapping of the scalar product  $g_n'(\mathfrak{A}^n, \mathfrak{v}^n)$ , induced by the mapping  $V$  given by formula (2.2). Thus our point of view does not contradict the well known results of Ogievetskij and Polubarinov<sup>15</sup> on the vector character of the interaction fields.

## 6. Mass as a Manifestation of Metrical Properties of the Manifold Associated with the Space of Observations

Among the physical quantities characterizing the particle only the mass has remained until now as a parameter without connections with geometrical properties of the manifold  $\mathbf{M}$  in question. To do this it is appropriate to follow the lines of general re-

lativity and treat the particle as a source of the gravitational field produced in the manifold  $\mathbf{N}$  and observed in  $\mathbf{M}$ .

Here a natural question arises whether as  $\mathbf{N}$  we should take  $\mathbf{N}_e, \mathbf{N}_n$ , or some other general Riemannian manifold. Since the "effective" particle whose manifestation is observed within  $\mathbf{M}$  is the topological sum — the union of supports of  $\mathbf{N}_e$  and  $\mathbf{N}_n$ , it seems reasonable to take  $\operatorname{supp} \mathbf{N} = \operatorname{supp} \mathbf{N}_e \cup \operatorname{supp} \mathbf{N}_n$  and to endow it with a suitable topology  $\operatorname{top} \mathbf{N}$ , the natural maximal atlas  $\operatorname{atl} \mathbf{N}$ , and a general Riemannian tensor field  $g'$  which will be specialized later on. Thus may proceed to the problem of determining the mass by means of the already chosen manifold  $(\operatorname{top} \mathbf{M}, \operatorname{atl} \mathbf{M})$  and its pseudo-riemannian metric  $g$  as well as the metric  $g'$  which remain for our disposal as parameters. Arguing as in the case of electromagnetic four-current we arrive at the results which may be formulated as follows.

Consider the standard four-dimensional manifold  $\tilde{\mathbf{E}}$  with  $\operatorname{supp} \tilde{\mathbf{E}} = \operatorname{supp} \mathbf{M} \cup \operatorname{supp} \mathbf{N}$ , endowed with the Minkowski metric. It is clear that in order to assure that  $\tilde{\mathbf{E}}$  permits unquantized mass it is sufficient to take, as before,  $\operatorname{supp} \tilde{\mathbf{E}}$  in the form of Cartesian product of the real line and either a  $k$ -pierced three-sphere or a three-torus and — on the other hand — any space  $\tilde{\mathbf{E}}$  in question may be supposed to be given in such a form. In order to agree with Sect. 2 we have to drill in the resulting manifold also the additional nonintersecting holes corresponding to the particles in question, so we have to consider the submanifold  $\mathbf{E}$  of  $\tilde{\mathbf{E}}$  with  $\operatorname{supp} \mathbf{E} = \operatorname{supp} \mathbf{M}$ .

Suppose now that  $\mathbf{M}$  permits quantized mass, i. e.  $4\pi(m_1 \tilde{R}_1 + \dots + m_l \tilde{R}_l)$  is in  $C_0(\mathbf{M})$  and that  $\mathbf{E}$  permits unquantized mass. Here  $m_1, \dots, m_l$  are quantized point masses, while  $\tilde{R}_1, \dots, \tilde{R}_l$  are the points of  $\mathbf{M}$  over which there are distributed the values  $\mathbf{p}_1^2 = 0, \dots, \mathbf{p}_l^2 = 0$  of momenta associated with  $m_1, \dots, m_l$ , respectively. The requirement for all  $\mathbf{p}_j^2$  to vanish is justified by the energy-momentum conservation law and can be assured by a proper choice of  $g$  and  $g'$ . In this case there exists a  $C^1$ -diffeomorphism  $\tilde{\Omega}: \mathbf{E} \rightarrow \mathbf{M}$  such that the mapping  $\Omega_*: \tilde{C}_2(\mathbf{E}) \rightarrow \tilde{C}_2(\mathbf{M})$  induced by  $\tilde{\Omega}$  satisfies the conditions

$$\int_{\Omega_* c_j} \tilde{\psi} = 4\pi m_j, \quad (6.1)$$



where  $\tilde{\psi}$  is in  $F^2(\mathbf{M})$  and corresponds to the energy-momentum four-tensor, and  $c_j$ ,  $j=1, \dots, \beta_2$ , are — as before — the fundamental cycles in  $\tilde{C}_2(\mathbf{E})$ . Here we have to suppose that  $l < \beta_2$ . Conversely, let  $\mathbf{M}$  be associated with an  $\mathbf{E}$  which permits unquantized masses by means of a  $C^1$ -diffeomorphism  $\Omega: \mathbf{E} \rightarrow \mathbf{M}$ . Then Eq. (6.1) is a necessary condition for  $\mathbf{M}$  to exhibit quantized masses as a manifestation of its topology.

In our case this topology is already determined when considering the electromagnetic and nuclear fields, so one is led to verify that for all elementary particles existing in reality Eq. (6.1) is fulfilled automatically. On the other hand this equation yields the allowable mass quanta. Next we can also find other conditions that restrict  $g$  and  $g'$ . Proceeding analogously to the case of electromagnetic four-current, we write  $\tilde{\psi}$  in local coordinates:  $\tilde{\psi} = T_{jk} dx^j dx^k$ , where  $T_{jk}$  represents the energy-momentum four-tensor, and define  $\tilde{\psi}^*$  as the form which can be written in the same local coordinates as  $\tilde{\psi}^* = *T_{jk} dx^j dx^k$ , where  $*T_{jk}$  represents the dual of  $T_{jk}$ . Then

$$\int_{\Omega_* c_j} \tilde{\psi}^* = 0 \quad (6.2)$$

give the required extra conditions.

Finally, let  $\delta$  be the mass density in  $\mathbf{M}$ , i. e.

$$\delta = (1/c) (d_1 dx^2 \wedge dx^3 + d_2 dx^3 \wedge dx^1 + d_3 dx^1 \wedge dx^2) \wedge dx^0 - \eta dx^1 \wedge dx^2 \wedge dx^3, \quad (6.3)$$

where the vector  $\mathbf{d} = (d_1, d_2, d_3)$  and the scalar  $\eta$  satisfy the continuity equation

$$\text{div } \mathbf{d} + (\partial/\partial x^0) \eta = 0.$$

Then the counterpart of the condition (4.4) reads:

$$H_\delta^3(\mathbf{M}) = 0 \quad (6.4)$$

and one is led to verify that for all elementary particles existing in reality Eq. (6.4) is fulfilled automatically.

Thus we conclude the present section with the statement that in our interpretation quantized point masses are situated, in general, at points  $\tilde{R}_1, \dots, \tilde{R}_l$  that are different from the points  $\tilde{P}_1, \dots, \tilde{P}_n$  corresponding to magnetic monopoles as well as  $\tilde{Q}_1, \dots, \tilde{Q}_m$  corresponding to nuclear monopoles. The topology on  $\mathbf{N}$  is induced from  $\mathbf{N}_e$  and  $\mathbf{N}_n$ , since  $\text{supp } \mathbf{N} = \text{supp } \mathbf{N}_e \cup \text{supp } \mathbf{N}_n$  and of course both topologies have to be compatible on  $\text{supp } \mathbf{N}_e \cap \text{supp } \mathbf{N}_n$ .

## 7. Equation of State as a Manifestation of Metrical Properties of the Manifold Associated with the Space of Observations

We have still some freedom in specifying the metrical properties of the manifolds in question. On the other hand, usually the equation of state for an elementary particle is considered in the Minkowski metric and this does not explain, in general, properties of this particle, e. g. the anomalous magnetic momenta of nucleons.

In view of Sect. 2 we may try to construct a consistent theory of such a particle choosing properly the metric  $g$  of the pseudo-riemannian manifold  $\mathbf{M}$ , basing on interactions yielded by the metric  $g'$  of the general Riemannian manifold  $\mathbf{N}$  and also by the mapping  $V$  given by formula (2.2). According to Sect. 3 we are led to two different manifolds  $\mathbf{N}$ : the “electromagnetic” manifold  $\mathbf{N}_e$  and “nuclear”  $\mathbf{N}_n$ , and when considering both fields: electromagnetic and nuclear together we have to utilize formula (3.1). Finally, owing to Sect. 4, 5, and 6, we have a theoretical way of determining the electric and nuclear charges  $e_j, g_j$ , and charges  $\mu_j, \nu_j$  due to the magnetic and nuclear monopoles as well as the point masses  $m_j$ .

Thus we are in a position to construct the pseudo-riemannian metric  $g$  effectively by means of known physical parameters  $e_j, g_j, \mu_j, \nu_j, m_j$ , being a manifestation of some curved space-time geometry in order to agree with the required properties of the particles in question, e. g. the anomalous magnetic momenta of nucleons or, more generally, of particles.

According to the considerations of Sect. 2, for a free particle we choose the mapping  $V$ , given by formula (2.2), so that the Hamiltonian  $\mathbf{H}$ , given by formula (2.4), had the form of the Dirac Hamiltonian, determined in terms of the metric  $g$ , i. e.

$$\mathbf{H} = c(g_{\alpha\alpha} \gamma^\alpha p^\alpha + g_{00} \gamma^0 m c), \quad (7.1)$$

where  $\gamma^\alpha$  denote the Dirac matrices and  $c$  is the light velocity in the vacuum. In this way we fix the mapping  $V$ . Next we transform the Hamiltonian  $\mathbf{H}$ , given by formula (3.1), which includes interactions with external fields, from  $\mathbf{H}(\mathbf{B}_\mathbf{N})$  to  $\mathbf{H}(\mathbf{B}_\mathbf{M})$ . Consequently we get

$$\mathbf{H} = c \left[ g_{\alpha\alpha} \gamma^\alpha \left( p^\alpha - \frac{1}{c_u} e A_e^\alpha - \frac{1}{c_v} g A_n^\alpha \right) + g_{00} \gamma^0 \left( m c - \frac{1}{c_e} e A_e^0 - \frac{1}{c_g} g A_n^0 \right) \right], \quad (7.2)$$



where  $e$  and  $g$  are the unit electric and nuclear charges, respectively,  $\mathfrak{A}_e$  denotes the four-vector potential corresponding to electromagnetic interactions within  $\mathbf{M}$ ,  $\mathfrak{A}_n$  denotes the four-pseudo-vector potential corresponding to nuclear interactions within  $\mathbf{M}$ , while  $c_e, c_g, c_\mu, c_\nu$  are the velocities of propagation of interactions  $A_e^0, A_n^0, (A_e^1, A_e^2, A_e^3), (A_n^1, A_n^2, A_n^3)$  within  $\mathbf{M}$ , respectively. It should be remarked here that  $\mathfrak{A}_e$  and  $\mathfrak{A}_n$  may, in general, depend on the local coordinates of  $\mathbf{M}$  and are determined by the mapping  $V$  together with the metrics  $g_e'$  and  $g_n'$ .

We may divide now the elementary particles into two groups according to their electric charge: charged and uncharged particles, and — independently — into two groups according to their nuclear charge: strongly interacting particles, i. e. endowed with a nuclear charge  $g_1 \neq 0$ , and particles which do not interact strongly, i. e. such that  $g_1 = 0$ . For a charged particle we put  $c_e = c_\mu$  since in this case interactions of electric and magnetic fields propagate with the same velocity, while for an uncharged particle we put  $c_e = \infty$  since there is no space of observations  $\mathbf{M}$ , when one could observe an electrostatic interaction for the same particle. An analogous consideration leads to the conclusion that we have to put  $c_g = c_\nu$  for a strongly interacting particle and  $c_g = \infty$  for a particle which does not interact strongly.

Taking into account the fact that in the Minkowski space-time the velocity of propagation of interactions coincides with the light velocity in the vacuum, we put as the scale  $c_\mu = c$ . Next, since in the Minkowski space-time the rest energy of a particle of the mass  $m_1 = m$  is  $m c^2$ , we calibrate  $g_{00} = 1$  provided that we ignore the gravitational interactions. Thus we are led to the problem of finding the four unknowns  $g_{11}, g_{22}, g_{33}$ , and  $c_\nu$  by means of the values of  $e_1, g_1, \mu_1$ , and  $\nu_1$ , given either theoretically by formulae (4.1), (5.1), (4.2), and (5.2), respectively, or referred to the experiment. Since in the already derived state equation we ignore the gravitational interactions, formulae (6.1) and (6.2) are fulfilled independently of the calculated metric  $g$ .

Let us consider, as an example, the electromagnetic properties of a particle  $\chi$ , confining ourselves to the interactions of  $\chi$  with the electromagnetic field and assuming that in this case the mapping  $V$  leads to a constant metric  $g$  with  $g_{11} =$

$g_{22} = g_{33}$ . Therefore the Hamiltonian (7.2) becomes

$$H = c \left[ g_{11} \sum_{\alpha=1}^3 \gamma^2 \left( p^2 - \frac{1}{c} e A_e^2 \right) + \gamma^0 \left( m c - \frac{1}{c} e A_e^0 \right) \right]$$

for a charged particle, while

$$H = c \left[ g_{11} \sum_{\alpha=1}^3 \gamma^2 \left( p^2 - \frac{1}{c} e A_e^2 \right) + \gamma^0 m c \right]$$

for an uncharged particle. The remaining unknown  $g_{11}$  may be referred to the experimental value of the magnetic moment  $M$ :

$$g_{11} = M, \quad (7.3)$$

where  $M$  is counted in the Bohr magneton units, e. g. equals 1.002 for the electron, 2.793 for the proton, and -1.913 for the neutron. It is clear that Eq. (7.3) may be considered as a *geometrical definition of the magnetic moment* in  $\mathbf{M}$ .

In order to obtain an analogous geometrical definition of the nuclear moment in  $\mathbf{M}$  one should abandon the hypothesis  $g_{11} = g_{22} = g_{33}$  assumed above and consider the general problem with four unknowns.

When assuming the electromagnetic field to vanish we see that the Hamiltonian of a free particle, with the physical properties determined in the preceding sections by geometrical properties of the manifold  $\mathbf{M}$ , becomes

$$H = c(\gamma \mathbf{p} + \gamma^0 m c) + c(M - 1) \gamma \mathbf{p}, \quad (7.4)$$

where the first addend corresponds to the *free Dirac equation*, while the second one is the Hamiltonian of *interactions* with forces analogous to the Coriolis forces (cf. Sect. 2), yielded by metrical properties of the manifold  $\mathbf{M}$ . On putting  $M = 1$ , i. e. considering the classical example of the Dirac electron, we obtain a description of the particle with help of the wave equation in the Minkowski space-time, since in this case we have  $g_{11} = M = 1$ , i. e. we have the Dirac equation.

This additional interaction can be taken into account differently by scaling over the mass and the light velocity to the *effective values* of these quantities which appear in the Minkowski space-time, namely

$$c M = c^* \quad \text{and} \quad m c^2 = m^* c^{*2}.$$

Then the solutions of the equation corresponding to the Hamiltonian (7.4) are solutions of the Dirac equation in the Minkowski space-time for the effective values of parameters determining the

particle, i.e.  $m^*$  and  $c^*$ . Thus if we ignore a small correction for the magnetic moment of the *electron*, this particle is really described by the standard Dirac equation, while the *proton* is a particle described by the same Dirac equation, but in a space-time with the light velocity and mass

$$c^* = 2.793 c \quad \text{and} \quad m^* = 0.1282 m,$$

respectively. In this interpretation the *neutron* behaves as a particle with a negative velocity: we have

$$c^* = -1.913 c \quad \text{and} \quad m^* = 0.2732 m.$$

## 8. Concluding Remarks

In the preceding sections we have presented a proposal of formulating the theory of elementary particles in terms of pure geometry: manifolds and their mappings. In the general formulation there still remain some quantities, mappings, and tensors left to be determined empirically in correspondence to different possibilities admissible within the theory. First of all we have to include among them the mapping  $V$ , given by formula (2.2), which has been strictly determined in this paper by assuming the shape of the Hamiltonian (7.1) for a free particle in the space of observations. This choice of the Hamiltonian is, as it is usually managed in physics, suggested by experiments.

The problem may be considered from a more general point of view by an adaptation of the transformation of Tharrats, Cercell, and Rojo<sup>16</sup> or its generalizations<sup>17,18</sup> leading in case of the nucleon interactions to an *effective potential* whose appearance in Eq. (2.3) is equivalent to expansions of the perturbation series in the quantum field theory:

$$V = \exp[i \sigma(\mathbf{p}/|\mathbf{p}|) \theta(|\mathbf{p}|/mc)], \quad (8.1)$$

where  $\sigma$  is the generator of the unitary rotation operator while  $\theta$  is the angle of counter-clockwise rotation of the coordinate axes in  $\mathbf{H}(B_N)$  around the direction of the unit vector  $\mathbf{p}/|\mathbf{p}|$ . More exactly, in Ref.<sup>18</sup>

$$\theta(|\mathbf{p}|/mc) = \arctan(|\mathbf{p}|/mc).$$

The effective potential obtained with the help of this transformation is equivalent to the potential of the quantum field theory, calculated up to the fourth order of the perturbation theory. An analogous remark concerns  $N_e$  and  $N_n$ .

Adapting the above procedure to our purposes we define the mapping  $V$  of  $\mathbf{H}(B_N)$  into  $\mathbf{H}(B_M)$  by formula (8.1), interpreting differently the rotation operator appearing there, namely, as the *spin operator* of the particle, which is a manifestation of the angular momentum of the particle in the space of observations  $\mathbf{M}$ , whereas this momentum is not observable in the own spaces  $N_e$  and  $N_n$ , since these spaces are constantly related to the particle.

The *isotopic spin* may now be interpreted analogously, as a property of the own spaces  $N_e$  and  $N_n$  into the space of observations  $\mathbf{M}$ . For instance, in the case of a nucleon, if we consider its nuclear structure only, the manifold  $\mathbf{M}$  is determined by a mapping of  $N_n$  alone and therefore is the same for the neutron as well as for the proton. If we take into account also the electromagnetic structure, we are led to determining  $\mathbf{M}$  not only by a mapping of  $N_n$ , but also of  $N_e$ . Since the manifold  $N_e$  is different for the neutron and for the proton, then also the manifold  $\mathbf{M}$  is different for these particles, if we consider its electromagnetic structure (cf. Section 4). Distinction between these spaces of observations is an effect equivalent to the introduction of the notion of isotopic spin.

The next problem which comes naturally is that of composing the electromagnetic and nuclear fields corresponding to two or more particles. Let us note that the determination of electric and nuclear charges as well as of magnetic and nuclear dipoles by the topological properties of  $\mathbf{M}$  did not in fact depend on the number particles. Thus, differently speaking, the topology of the space of observations is already determined also in the case of many particles, namely, by the conditions (4.1), (5.1), (4.2), (5.2) as well as by the homology conditions (4.4) and (5.4).

Similarly, the conditions (6.1) and (6.2), connected with the mass and momenta, as well as the corresponding homology condition (6.4) gave us restrictions for the pseudo-riemannian metric  $g$  and the general Riemannian metric  $g'$  of  $\mathbf{N}$  with  $\text{supp } \mathbf{N} = \text{supp } N_e \cap \text{supp } N_n$ , independently of the number of particles. An extrapolation of this regularity leads to a natural suggestion to postulate that the metric  $g$ , determined in Sect. 7 in the case of one particle, provided that we ignore the gravitational interactions, be expressed in the same way by  $M$  in the case of many particles, where  $M$  de-

notes now the magnetic moment of the *system* of particles, counted in the Bohr magneton units.

In the general case, i. e. when we do not ignore the gravitational interactions, greater freedom in determining the metric  $g$  may be utilized in order to explain other physical parameters of the particle (or of a system of particles) as the quadrupole moment, the coupling constants, the spin-orbit coupling constant, etc. In consequence this should lead to some rules of choice for the topology and metrics of the pseudo-riemannian manifolds which correspond to the really existing systems of particles.

Now we arrive at a possibility of applying directly this concept not only in order to determine the pseudo-riemannian manifold  $\mathbf{M}$  corresponding to the system of particles in question and the associated general Riemannian manifolds  $\mathbf{N}_e$  and  $\mathbf{N}_n$ , but also in order to determine some *selection rules for a nuclear reaction*. Here, by a nuclear reaction we have to mean an *admissible* mapping of the system of manifolds corresponding to the reacting particles onto the system of manifolds corresponding to the particles obtained as the effect of this reaction so that the manifold  $\mathbf{M}$  associated with the whole system would remain *unchanged*.

Thus, if all the reacting particles are included in the system under consideration, the invariance of the corresponding manifold  $\mathbf{M}$ , which concerns

in particular the light cone (beyond possible singularities) suggests the use of *conformal mappings*<sup>19</sup>. In the case where an external particle is introduced, the light cones do not need any more to be invariant. Nevertheless, owing to a finite velocity with which the external particle in question approaches, the light cones change in a bounded way, and this suggests the use of *quasiconformal mappings* which have the required property<sup>9</sup>. A generalization of known plane results restricting the domains admissible for these mappings<sup>20</sup> would lead to criteria for mappings interpreted as nuclear reactions, i. e. *admissible* for such an interpretation.

Finally, the proposed formulation of the theory of von Westenholz<sup>2</sup> should lead in our interpretation of manifolds associated with a particle to determining the distribution of charges inside a neutral particle; concerning this distribution it is known that it implies the vanishing dipole moment<sup>21</sup>.

At the end, the authors should like to emphasize that the proposed concept has a general character and the decision about its correctness may be assured by its development only.

In this place the authors should like to thank their Colleagues Profs. T. Tietz and W. Tybor (Łódź), and L. S. Woronowicz (Warsaw) as well as Dr. J. Komorowski (Warsaw) for reading the manuscript and critical remarks.

<sup>1</sup> R. L. Bishop and S. I. Goldberg, *Tensor Analysis on Manifolds*, Macmillan, New York—London 1968.

<sup>2</sup> C. von Westenholz, *Ann. Inst. H. Poincaré Sect. A (N.S.)* **15**, 189 [1971].

<sup>3</sup> C. W. Misner and J. A. Wheeler, *Ann. Physics* **2**, 525 [1957].

<sup>4</sup> P. A. M. Dirac, *Proc. Roy. Soc. London Ser. A* **322**, 435 [1971].

<sup>5</sup> J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields*, McGraw-Hill, New York 1965.

<sup>6</sup> R. Narasimhan, *Analysis on Real and Complex Manifolds*, Masson and North-Holland, Paris-Amsterdam 1968. — L. Sario and K. Oikawa, *Capacity Functions*, Springer, Berlin 1969.

<sup>7</sup> S. S. Chern, *Complex Manifolds without Potential Theory*, Van Nostrand, Princeton (N. J.) 1967.

<sup>8</sup> A. Gão, *Arquivo Inst. Gulbenkian Ci. A Estud. Mat. Fis.-Mat.* **5**, 1, 1 [1967].

<sup>9</sup> J. Ławrynowicz, *Rep. Math. Phys.* **5**, 203 [1974].

<sup>10</sup> P. A. M. Dirac, in *Proc. Internat. Conf. of Relativistic Theories of Gravitation*, Warsaw and Jabłonna 1962, ed. by L. Infeld, Pergamon-Press, Gauthier-Villars and PWN, Warsaw 1964, p. 163. — P. Burcev, *Czechoslovak J. Phys. B* **23**, 1172 [1973]. — L. Halpern, *Z. Naturforsch.* **28a**, 1613 [1973].

<sup>11</sup> N. E. Steenrod, *The Topology of Fibre Bundles*, Princeton Univ. Press, Princeton (N. J.) 1951. — J. Eells Jr., *Fibrations and Fibre Bundles*, Univ. of Ghana Lecture Notes, Legon 1966. — A. Trautman, *Rep. Math. Phys.* **1**, 29 [1970].

<sup>12</sup> P. A. M. Dirac, *Phys. Rev.* **74**, 817 [1948].

<sup>13</sup> E. H. Spanier, *Algebraic Topology*, McGraw-Hill, New York 1966.

<sup>14</sup> G. de Rham, *Variétés différentiables. Formes, courants, formes harmoniques*, Hermann, Paris 1955.

<sup>15</sup> V. I. Ogievetskij and I. V. Polubarinov, *Ann. Physics* **25**, 358 [1963].

<sup>16</sup> J. Tharrats, O. Cercell, and O. Rojo, *J. Math. Phys.* **6**, 1315 [1965].

<sup>17</sup> J. Florczak, *Acta Phys. Polon.* **34**, 833 [1968].

<sup>18</sup> J. Florczak, *Bull. Soc. Sci. Lettres Łódź*, to be published.

<sup>19</sup> R. Penrose, *J. Math. Phys.* **8**, 345 [1967]. — R. Penrose, *J. Theor. Phys.* **1**, 61 [1968]. — J. Ławrynowicz and W. Waliszewski, *Math. Scand.* **28**, 45 [1971]. — W. Kopyczyński and L. S. Woronowicz, *Rep. Math. Phys.* **2**, 35 [1971].

<sup>20</sup> B. V. Šabat, in *Certain Problems of Mathematics and Mechanics*, On the occasion of the seventieth birthday of M. A. Lavrent'ev, ed. by N. N. Bogoljubov, Izdat. "Nauka", Leningrad 1970 [in Russian], p. 251.

<sup>21</sup> W. Tybor, private communication.